OCR
RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2602/1

Pure Mathematics 2

Wednesday **12 JANUARY 2005** Afternoon 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer **all** questions.
- You are permitted to use only a scientific calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 60.

This question paper consists of 5 printed pages and 3 blank pages.

1 **(a)** Fig. 1 shows the graph of $y = \mathrm{f}(x)$ with domain $-1 \leqslant x \leqslant 1$.

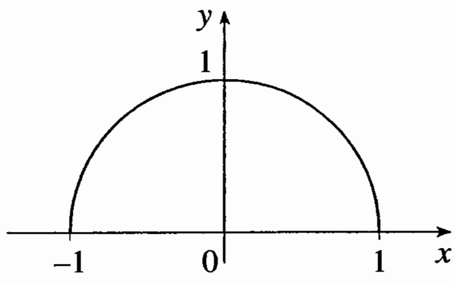


Fig. 1

Sketch the graphs of

(i) $y = -\mathrm{f}(x)$,

(ii) $y = 2\mathrm{f}(x-1)$. [3]

(b) Using a suitable substitution, or otherwise, find $\int x(x^2+1)^{10}\,\mathrm{d}x$. [4]

(c) Differentiate $\ln\left(1+\frac{1}{x}\right)$, simplifying your answer. [4]

(d) Given that $y = x\mathrm{e}^{2x}$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

Hence verify that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 0.$$ [5]

[Total 16]

2 A factory makes widgets. From January 2006, the production manager plans to phase out the production of old widgets and phase in production of new widgets. He models this process as follows.

For old widgets, the monthly production will form a geometric sequence with common ratio 0.9. The production in month 1 (January 2006) will be 5000.

(i) Find the production figures predicted by the model for months 2, 3 and 12. [4]

(ii) Find the *total* production of old widgets for the 24 months from the start of January 2006. [3]

(iii) Show that the total production of old widgets from the start of January 2006 will not exceed 50 000. [2]

For new widgets, the production in month 1 (January 2006) will be 500. Production will increase by 10% per month, forming a geometric sequence.

(iv) Write down an expression for the production of new widgets in month n. Hence show that monthly production of new widgets will first exceed that of old widgets in month k, where k is the smallest integer for which

$$\left(\frac{11}{9}\right)^{k-1} > 10.$$

Find this value of k. [6]

[Total 15]

3 Fig. 3 shows the graph of $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = ax - x^{\frac{2}{3}}, \qquad x \geqslant 0.$$

The curve crosses the x-axis at the point $\mathrm{A}(\frac{1}{8}, 0)$ and has a turning point at B.

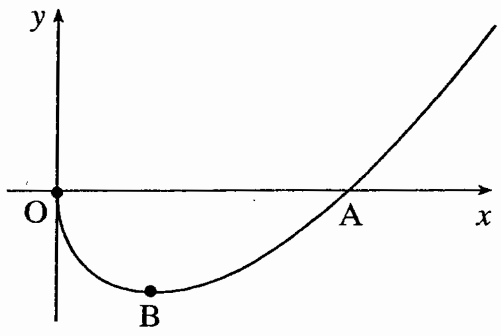


Fig. 3

(i) Show that $a = 2$. [2]

(ii) Find as an exact fraction the gradient of the curve at A. What happens to the gradient of the curve near the origin? [4]

(iii) Find the exact coordinates of B. State the range of the function $\mathrm{f}(x)$. [5]

(iv) Calculate the area of the region enclosed by the curve and the x-axis. [4]

Total [15]

4 A pan of water is heated. The temperature T°C of the water t minutes after switching on the heat is modelled by the equation

$$T = 105 - 85e^{-t}.$$

(i) Using this model, calculate the initial temperature of the water, and the initial rate of temperature increase. [3]

(ii) Find the time predicted by the model for the water to reach its boiling point of 100 °C. [3]

Once the water reaches 100 °C, the heat is switched off. The temperature T°C of the water is now modelled by the equation

$$T = 20 + ae^{-kt},$$

where t now denotes the time in minutes after the heat is switched off.

(iii) Write down the value of a. What does the constant 20 in the equation represent? [2]

(iv) Show that plotting values of $\ln(T - 20)$ against t would result in a straight line. [2]

Fig. 4 shows the graph of $\ln(T - 20)$ against t.

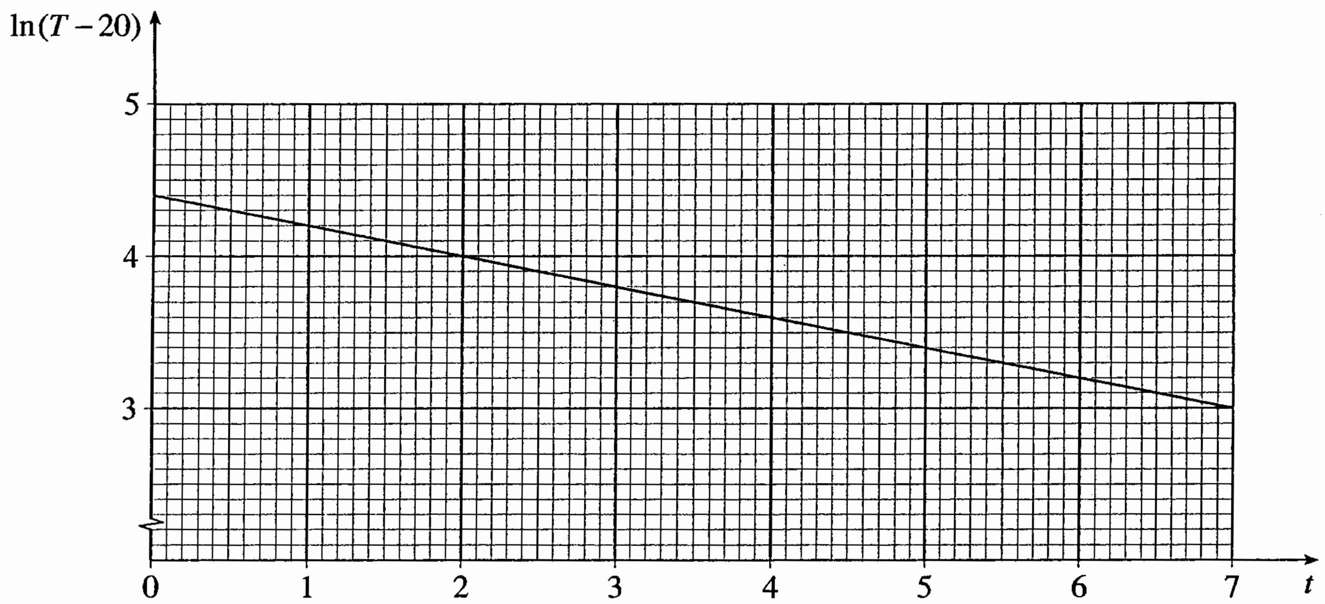


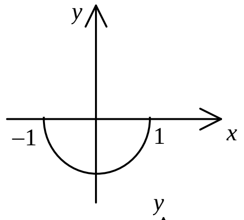
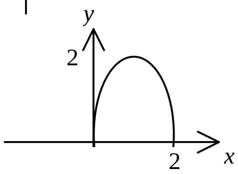
Fig. 4

(v) Use the graph to estimate the value of k, and the time for the temperature of the water to drop from 100 °C to 50 °C. [4]

[Total 14]

Mark Scheme

P2 (2602)Jan 2005 Final mark scheme

1 (a) (i) 	B1	
(ii) 	B1 B1 [3]	Translation 1 unit to right Stretch in y direction s.f. 2
(b) $\int x(x^2+1)^{10}dx$ let $u = x^2 + 1$, $du = 2x\ dx$ $= \int \frac{1}{2}u^{10}du$ $= \frac{1}{22}u^{11}+c$ $= \frac{1}{22}(x^2+1)^{11}+c$	M1 A1 A1 A1	Substituting $u^{10} =(x^2 + 1)^{10}$ in integral $\int \frac{1}{2}u^{10}du$ [condone no du] $\frac{1}{22}u^{11}$ oe – allow $\frac{1}{2}\times\frac{1}{11}u^{11}$ or $\frac{1}{2}\times\frac{1}{11}(x^2+1)^{11}$ cao – must have + c and $\frac{1}{22}$. Allow ½ c.
or $= \frac{1}{22}(x^2+1)^{11}+c$ by inspection	M1 A1 A1 A1 [4]	$k(x^2 + 1)^{11}$ $k = \frac{1}{2}\times\frac{1}{11}=\frac{1}{22}$ $\frac{1}{22}(x^2+1)^{11}$ $+c$ or ½ c
(c) $y=\ln(1+\frac{1}{x})$ let $u = 1+\frac{1}{x}$, $\frac{du}{dx}=-\frac{1}{x^2}$ $y = \ln u \Rightarrow \frac{dy}{du}=\frac{1}{u}$ $\Rightarrow \frac{dy}{dx}=\frac{1}{u}.(-\frac{1}{x^2})=\frac{1}{1+\frac{1}{x}}.(-\frac{1}{x^2})$ $= -\frac{1}{x^2+x}$ $= -\frac{1}{x(x+1)}$ *or* $y=\ln(1+\frac{1}{x})=\ln(\frac{x+1}{x})=\ln(x+1)-\ln x$ $\Rightarrow \frac{dy}{dx}=\frac{1}{x+1}-\frac{1}{x}$	B1 B1 M1 A1cao M1 A1,A1 A1cao [4]	$\frac{dy}{du}=\frac{1}{u}$ or $\frac{1}{1+\frac{1}{x}}$ seen $\frac{d}{dx}(\frac{1}{x})=-\frac{1}{x^2}$ seen $\frac{dy}{du}\times\frac{du}{dx}$ denominator simplified; mark final answer; accept $-\frac{1}{x^2+x}$ as final answer [not $-\frac{x^{-2}}{1+x^{-1}}$] $\frac{1}{x+1}$, $\frac{1}{x}$; no need to combine, but mark final answer.
(d) $y = x\ e^{2x}$ $\Rightarrow$ $dy/dx = e^{2x} + 2x\ e^{2x}$ $\Rightarrow$ $d^2y/dx^2 = 2\ e^{2x} + 2\ e^{2x} + 4x\ e^{2x}$ $= 4\ e^{2x} + 4x\ e^{2x}$ $\Rightarrow \frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y = 4e^{2x} + 4xe^{2x} - 4e^{2x} - 8xe^{2x} + 4xe^{2x} = 0.$	B1 B1 M1 B1cao E1	$2x\ e^{2x}$ …+ e^{2x} Finding their d^2y/dx^2 and attempting to verify www (NB fortuitous result with

	[5]	$dy/dx = 2xe^{2x}$ and $d^2y/dx^2 = 4x\ e^{2x}$ – SC2)

2 (i) 4500, 4050 $5000 \times 0.9^{11} = 1569$	B1,B1 M1A1 [4]	SC B1 for 4050, 3645 or 1570 – must be rounded, but penalise once only. Allow M1A0 for 5000×0.9^{12} (= 1412.14…)
(ii) $\frac{5000(1-0.9^{24})}{(1-0.9)}$ or $\frac{5000(0.9^{24}-1)}{(0.9-1)}$ = 46 011.67… = 46 012	M1 A1 A1 [3]	sum formula – correct expression correctly evaluated correctly rounded - accept 46 000 or 46 010 or 46 011 – penalise lack of rounding once only (see 2(i))
(iii) $S_\infty = \frac{a}{1-r}$ $= \frac{5000}{1-0.9} = 50\,000$ [So total production can't exceed 50 000]	M1 A1	Sum to infinity used, [or $S_n = \frac{5000(1-0.9^n)}{(1-0.9)}$ together with $0.9^n \to 0$] 50 000 [or $\to$ 50 000]
or $S_n = 50000(1-0.9^n) < 50\,000$	B2 [2]	
(iv) $500 \times 1.1^{n-1}$ $\Rightarrow 500 \times 1.1^{k-1} > 5000 \times 0.9^{k-1}$ $\Rightarrow \frac{1.1^{k-1}}{0.9^{k-1}} > \frac{5000}{500}$ [$\Rightarrow \left(\frac{1.1}{0.9}\right)^{k-1} > 10$] $\Rightarrow \left(\frac{11}{9}\right)^{k-1} > 10$ *	B1 M1 E1 (3)	 forming inequality – allow equality [condone n instead of k, or k instead of $k-1$] if working from equality, final inequality must be adequately explained
$\Rightarrow (k-1)\ln(11/9) > \ln 10$ $\Rightarrow k > 1 + \frac{\ln 10}{\ln(11/9)} = 12.47\ldots$ $\Rightarrow k = 13$	M1 A1 A1 (3)	taking lns or logs Re-arranging to find k or $k-1$ cao
or by trial and improvement: $\left(\frac{11}{9}\right)^{11} = 9.09 < 10$ $\left(\frac{11}{9}\right)^{12} = 11.11 > 10$ $\Rightarrow k = 13$	 B1 B1 B1dep (3)	 Or $n = 12 \Rightarrow$ old 1569, new 1426 Or $n = 13$, old = 1412, new 1569 dep both B1s SCB1: $k = 13$ unsupported

3 (i) f(x) = 0 when x = 1/8 $\Rightarrow \frac{1}{8}a - \left(\frac{1}{8}\right)^{2/3} = 0$ $\Rightarrow a = 8 \times ¼ = 2$ *	M1 E1	f(1/8) = 0 NB Answer given! $a = \frac{(1/8)^{2/3}}{(1/8)} = 2$ not enough: must show $(1/8)^{2/3} = 1/4$ or $(1/8)^{-1/3}$ or $\frac{1}{(1/8)^{1/3}}$ (oe)
or by verification: f(1/8) = 2×(1/8) – (1/8)$^{2/3}$ = ¼ – ¼ = 0	M1 E1 [2]	f(1/8) = 0, but must show some working as above
(ii) f(x) = 2x – $x^{2/3}$ $\Rightarrow$ f′(x) = 2 – (2/3)$x^{-1/3}$ When x = 1/8, f′(1/8) = 2 – (2/3)×(1/8)$^{-1/3}$ = 2 – 2/3 × 2 = 2/3 As $x \to 0$, gradient tends to (negative) infinity.	M1 A1 A1 A1dep [4]	Differentiating or $0.\dot{6}$ but must be exact – not 0.67 etc. allow ‘undefined’ or ‘vertical’, but not ‘very steep’ or ‘asymptotic’, dep negative power of x in their derivative. After 0 scored, SCB1 for correct derivative in (iii)
(iii) f′(x) = 0 when $2 - \frac{2}{3}x^{-1/3} = 0$ $\Rightarrow 2 = \frac{2}{3}x^{-1/3}$ $\Rightarrow x^{-1/3} = 3$ $\Rightarrow x = 1/27$ $y = 2 \times 1/27 - (1/27)^{2/3}$ $= \frac{2}{27} - \frac{1}{9} = -\frac{1}{27}$ $\Rightarrow$ range is $y \geq -\frac{1}{27}$	M1 A1ft A1cao A1cao B1ft [5]	Equating derivative to zero or better, e.g. $x^{1/3} = 1/3$ allow ft on their –1/3 unless eased or; $0.\dot{0}3\dot{7}$ must be exact, but allow recovery from logs if answer correct or $-0.\dot{0}3\dot{7}$ or f(x) $\geq -\frac{1}{27}$, ft their $-\frac{1}{27}$, Not $x \geq -\frac{1}{27}$, accept $\geq -\frac{1}{27}$
(iv) $A = (-)\int_0^{1/8}(2x - x^{\frac{2}{3}})dx$ $= (-)\left[x^2 - \frac{3}{5}x^{5/3}\right]_0^{1/8}$ $= (-)\left(\frac{1}{64} - \frac{3}{5} \times \frac{1}{32}\right)$ $= (-)\frac{1}{320}$ or (–) 0.003125	M1 B1 M1 A1 [4]	Correct integral and limits (soi) – allow $\int_{\frac{1}{8}}^{0}$ $x^2 - \frac{3}{5}x^{5/3}$ substituting limits (upper – lower) accept ±0.0031 or better.

4 (i) 105 – 85 = 20 °C $\frac{dT}{dt} = 85\,e^{-t}$ = 85 when $t = 0$	B1 M1 A1 [3]	$85\,e^{-t}$ M1 can be implied by $85e^0$ seen. 85 unsupported M0
(ii) $105 - 85\,e^{-t} = 100$ $\Rightarrow$ $85\,e^{-t} = 5$ $\Rightarrow$ $e^{-t} = 5/85 = 1/17$ $\Rightarrow$ $e^t = 17$ $\Rightarrow$ $t = \ln 17 = 2.83$ mins	M1 M1 A1cao [3]	Equating taking lns or logs correctly on correct eqn 2.8 or better B3 trial and improvement $\Rightarrow$ 2.83 or better
(iii) $a = 80$ 20 represents room or background temperature	B1 B1 [2]	or initial temperature from first part, or starting, original, minimum temperature
(iv) $T - 20 = a\,e^{-kt}$ $\Rightarrow \ln(T - 20) = \ln a - kt$ cf $y = c + mx$ so straight line with gradient $-k$ and intercept $\ln a$.	M1 A1ft [2]	taking lns correctly – ft their a Comparison with straight line equation, but if explicit must be correct. ft their value of a, allow $kt \ln e$
(v) gradient = $\frac{3-4}{7-2} = -0.2$. $\Rightarrow k = 0.2$	M1 A1	$k = -0.2$ M1A0
or using a point to find k: $\ln(T - 20) = \ln 80 - kt$ e.g. using (2, 4): $4 = 4.4 - k \times 2$ $\Rightarrow k = 0.2$	M1 A1	substituting cords of a point into equation – ft their a allow art 0.20 for k from correct working
When $T = 50$, $\ln(T - 20) = \ln 30 = 3.40$ reading from graph, $t = 5$	M1 A1	3.40 soi
or using equation or ln equation: $T = 20 + 80e^{-0.2t} = 50$ $\Rightarrow$ $80\,e^{-0.2t} = 30$ $\Rightarrow$ $e^{-0.2t} = 0.375$ $\Rightarrow$ $-0.2t = \ln(0.375)$ $\Rightarrow$ $t = \frac{\ln(0.375)}{-0.2} = 4.9$	M1 A1 [4]	Solving by taking lns or $\ln 30 = \ln 80 - 0.2t$ [$3.40 = 4.4 - 0.2t$] or $t = 5$ allow art 4.9 or 5.0

Examiner’s Report